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EXISTENCE OF PERMUTATION CYCLES AND MANIPULATION OF CHOICE FUNCTIONS*

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ABSTRACT

Let σ be a social preference function, and let $v(\sigma)$ be the Nakamura number of σ. If W is a finite set of cardinality at least $v(\sigma)$ then it is shown that there exists an acyclic profile P on W such that $\sigma(P)$ is cyclic. Any choice function which is compatible with σ can then be manipulated. A similar result holds if W is a manifold (or a subset of Euclidean space) with dimension at least $v(\sigma) - 1$.

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1. INTRODUCTION

It is well known that if σ is a "non-collegial" social preference function on a "universal" set of alternatives, W, then it is possible to construct an acyclic profile P on W such that $\sigma(P)$ is itself cyclic [1]. However, if W is restricted in some way, by cardinality when finite or by dimension when W is a vector space, then there may exist no profile P such that $\sigma(P)$ is cyclic. For example, consider a q-rule, σ, where any coalition with q people is winning. Ferejohn and Grether [2] showed that if W is finite of cardinality $|W| < \frac{n}{n-q}$ then $\sigma(P)$ cannot be cyclic when P is itself acyclic. Conversely if $|W| \geq \frac{n}{n-q}$ then an acyclic profile P can be constructed such that $\sigma(P)$ is cyclic. A parallel result by Greenberg [4] has shown that a <u>core</u> exists for $\sigma(P)$ on a convex set W if and only if the dimension of W is no greater than $\frac{q}{n-q}$. Recently [10, 11, 12, 13] Greenberg's theorem has been extended to the case of an arbitrary voting rule where the dimension bound of W is given in terms of the <u>Nakamura</u> [7] number $v(\sigma)$ of σ. The purpose of this paper is to extend the Ferejohn-Grether result by showing that $|W| < v(\sigma)$ is necessary and sufficient condition for the non-existence of cycles and the existence of a core, when W is finite and individual preferences are acyclic. Using this result it is shown that when $|W| \geq v(\sigma)$ then σ can be manipulated. Using earlier results it is also shown that when W is a manifold, or less generally a subset of Euclidean space, of dimension at least $v(\sigma) - 1$ then again σ can be manipulated.

2. STATEMENT OF RESULTS

Throughout the paper we shall use the definitions, notation and terminology of [8] to which the reader is referred. We review these definitions as follows.

When P is a binary relation on a set of alternatives, W, then we use the notation xPy to mean $(x,y) \in P$. A <u>strict preference</u> P on the set, W, is a binary relation on W which is <u>irreflexive</u> (i.e., xPx for no x in W) and <u>asymmetric</u> (i.e., xPy implies not(yPx) for any x,y in W). A preference, P, is <u>transitive</u> if xPy and yPz implies xPz, for any $x,y,z \in W$. A subset $\{x_1,\ldots,x_r\}$ of W is a <u>P-cycle</u> iff $x_1Px_2 \cdots Px_rPx_1$. If P is a preference and W contains some P-cycle then P is said to be <u>cyclic</u> on W. If W contains no P-cycle, then P is said to be acyclic on W. A <u>profile</u> for a society $N = \{1,\ldots,i,\ldots,n\}$ is a list $P = \{P_1,\ldots,P_n\}$ of strict preferences, one for each member of the society. Let B(W) represent the class of strict preferences on W, and $B(W)^N$ the class of profiles on W whose components are strict preferences. Similarly let A(W) and $A(W)^N$ represent the class of acyclic strict preferences and acyclic profiles. When $P_i \in B(W)$ then the indifference relation $I(P_i)$ associated with P_i is given by $xI(P_i)y$

iff neither xP_iy nor yP_ix. Weak preference $R(P_i)$ is defined by $xR(P_i)y$ iff not(yP_ix).

If P_i and P_j are both strict preferences then define the meet $P_i \wedge P_j \in B(W)$ by

$$x(P_i \wedge P_j)y \text{ iff } xP_iy \text{ and } xP_jy.$$

Clearly if xP_iy and yP_jx then $xI(P_i \wedge P_j)y$ where $I(P_i \wedge P_j)$ is the indifference relation associated with $P_i \wedge P_j$. Moreover, if P_i, P_j both belong to $A(W)$ then so does $P_i \wedge P_j$. In this paper the set of alternatives, W, will be assumed to be of finite cardinality, $|W| = w$, although an occasion we refer to results when W is a topological vector space of dimension $\dim(W)$.

A social preference function (SF) is a function, σ, which assigns to any profile, P, of strict preferences on a set W a strict preference, $\sigma(P)$, on W and moreover satisfies the independence axiom (see [9, Def. 2.2]). Here we shall principally examine simple social preference functions, or voting rules. We call a subset, M, of the society, N, a coalition. A coalition, M, is decisive for a SF, σ, iff for any profile, P, and any alternatives $x,y \in W$

$$xP_iy \text{ for all } i \in M \Rightarrow x\sigma(P)y.$$

Let $\mathbb{D}_\sigma$ refer to the class of decisive coalitions for σ. If σ is such that $x\sigma(P)y \Rightarrow xP_iy$ for all i in some coalition M in $\mathbb{D}_\sigma$, then σ is said to be simple, and is referred to as a voting rule. Suppose that for a social preference function, σ, there exists $M_1, M_2 \in \mathbb{D}_\sigma$ such

that $M_1 \cap M_2 = \Phi$. It is then possible to construct a profile P such that $xP_iy \ \forall i \in M_1$ and $yP_ix \ \forall i \in M_2$, if $x,y \in W$. In this case we obtain $x\sigma(P)y$ and $y\sigma(P)x$ so that $\sigma(P)$ is not asymmetric. We forbid this by assuming that σ is proper i.e., that $M_1 \cap M_2 \neq \Phi$ whenever $M_1, M_2 \in \mathbb{D}_\sigma$.

If σ is a voting rule whose decisive coalitions are $\mathbb{D}_\sigma = \{M \subset N : |M| \geq q\}$ then σ is called a q-rule. In this case we write σ_q and $\mathbb{D}_q$. For convenience we shall restrict the term q-rule to those cases where q is integer and $n/2 < q \leq n - 1$. An example of a q-rule is simple majority rule which is defined by taking $q = k + 1$ wherever n is odd and equal to $2k + 1$ or n is even and equal to $2k$.

An important question in social choice is whether an SF, σ, is acyclic in the sense that $\sigma(P) \in A(W)$ for all $P \in A(W)^N$ and appropriate W. If a point $x \in W$ belongs to a $\sigma(P)$-cycle then we shall say x belongs to the global cycle set $GC(\sigma,W,N,P)$. In the case that W is of finite cardinality, then $GC(\sigma,W,N,P)$ is empty only if the core (or global optima set)

$$GO(\sigma,W,N,P) = \{x \in W : \nexists y \in W \text{ st. } y\sigma(P)x\}$$

is non-empty. We shall write $GC(\sigma,P)$ and $GO(\sigma,P)$ for these two sets when there is no ambiguity. We also define the pareto set for coalition M and profile P by $GO(M,P) = \{x \in W : \nexists y \in W \text{ st. } yP_ix \ \forall i \in M\}$. Note, of course, that

$$GO(\sigma,P) \subset \bigcap_{\mathbb{D}_\sigma} GO(M,P)$$

with equality when σ is a voting rule.

Without imposing further restrictions on W or P, a <u>necessary</u> condition for $\sigma(P)$ to be acyclic is that $\mathbb{D}_\sigma$ be <u>collegial</u>. More formally, if $\mathbb{D} = \{A_1, \ldots, A_r\}$ is a class of coalitions then the intersection $K(\mathbb{D}) = A_1 \cap \cdots \cap A_r$ is called the <u>collegium</u> of $\mathbb{D}$. $\mathbb{D}$ is called collegial or non-collegial depending on whether $K(\mathbb{D})$ is non-empty or empty. If $K(\mathbb{D}_\sigma)$ is non-empty then the SF, σ, is called <u>collegial</u> and the members of $K(\mathbb{D}_\sigma)$ are known as <u>vetoers</u>; otherwise σ is called <u>non-collegial</u>. If W is a set of finite, but arbitrary, cardinality and σ is a non-collegial SF then it is always possible to find an acyclic profile P on W such that $\sigma(P)$ is cyclic [1].

Ferejohn and Grether [2] have in a sense refined this result, in the case of a q-rule, by showing that the set W of alternatives on which $\sigma(P)$ is cyclic must be of sufficiently high cardinality. More precisely, for σ a q-rule they showed that if W is of finite cardinality, $|W|$, with $|W| = w$ then $\sigma(P) \in A(W)$ for all $P \in A(W)^N$ iff $q > \left[\frac{w-1}{w}\right]n$. Thus the q-rule, σ, is <u>acyclic</u> iff $|W| < \frac{n}{n-q}$. In other words $\sigma(P)$ is acyclic if $|W| < \frac{n}{n-q}$, and an acyclic profile P can be found on W whenever $|W| \geq \frac{n}{n-q}$ such that $\sigma(P)$ is cyclic. The first inequality may also be written $|W| \leq v(n,q) + 1$ where $v(n,q)$ is the largest integer which is strictly less than $\frac{q}{n-q}$.

In a recent paper Greenberg [4] extended this result by showing that for a q-rule, σ, if W is compact, convex of dimension w, and each individual preference is continuous and convex then the core, $GO(\sigma,P)$, is non-empty iff $q > \left[\frac{w}{w+1}\right]n$. Again this inequality can be written $\dim(W) \leq v(n,q)$. Greenberg then gave an alternative proof of the Ferejohn-Grether result by embedding a finite set, W, of cardinality $(w + 1)$, in the w-dimensional simplex.

In two recent papers by Schofield [10] and Strnad [13] Greenberg's result has been extended to the case of an arbitrary non-collegial voting rule. More particularly Schofield [10, 11] and Strnad [13] independently showed that for any non-collegial rule there exists an integer $v(\sigma)$, called the <u>Nakamura number</u> with the following property: If W is compact, convex of dimension w, and preferences are continuous and convex then the core, $GO(\sigma,P)$ is non-empty iff $w \leq v(\sigma) - 2$. It is easy to show [10] that $v(\sigma) - 2 = v(n,q)$, which indicates that the Schofield-Strnad result generalizes Greenberg's theorem.

The purpose of this paper is to extend Greenberg's procedure to obtain a generalization of the Ferejohn-Grether theorem to the case of arbitrary voting rule. First of all we define the Nakamura number [7].

<u>Definition 1</u>: If $\mathbb{D}$ is a family of subsets of N, then the <u>Nakamura number</u> $v(\mathbb{D})$ of $\mathbb{D}$ is defined as follows:

(i) if $K(\mathbb{D}) \neq \Phi$ then $v(\mathbb{D}) = \infty$.

(ii) if $K(\mathbb{D}) = \Phi$ then

$$v(\mathbb{D}) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D} \text{ and } K(\mathbb{D}') = \Phi\}.$$

If σ is a social preference function with decisive coalitions $\mathbb{D}_\sigma$, then the <u>Nakamura number</u>, $v(\sigma)$, of σ is defined to be $v(\mathbb{D}_\sigma)$.

Since we always assume that σ is proper then $v(\sigma) \geq 3$. The principal theorem of this paper is as follows.

Theorem 1: Suppose that σ is a social preference function and W is a set of alternatives with finite cardinality $|W| = w$.

(i) if $w \geq v(\sigma)$ then there exists a profile $p \in A(W)^N$ such that $GO(\sigma,P)$ is empty and $GC(\sigma,P)$ is non-empty

(ii) if $w = v(\sigma)$ and $P \in A(W)^N$ such that $GC(\sigma,P)$ is non-empty then $GC(\sigma,P) = GO(N,P)$

(iii) if $w \leq v(\sigma) - 1$ and $P \in A(W)^N$ and in addition σ is a voting rule then $GC(\sigma,P)$ is empty and thus $GO(\sigma,P)$ is non-empty.

As we have noted, when σ_q is a q-rule, $v(\sigma_q) = v(n,q) + 2$ and this as a corollary we obtain the Ferejohn-Grether result that $\sigma_q(P)$ is acyclic iff $|W| \leq v(n,q) + 1$.

While Nakamura [7] introduced the notion of $v(\sigma)$ to obtain a version of parts (i) and (iii) of this theorem, the proof offered here is intrinsically geometric and constructive and is of some additional interest since it sheds some light on the structure of the "top" cycle set and on the possibility for coalition manipulation.

3. EXISTENCE OF CYCLES

The key to the proof of Theorem 1 is the notion of a representation of σ.

First of all for an arbitrary non-collegial family $\mathbb{D}$ of coalitions we shall call a subfamily $\mathbb{D}'$ of $\mathbb{D}$ a Nakamura family iff

(i) $|\mathbb{D}'| = v(\mathbb{D})$

(ii) $K(\mathbb{D}') = \Phi$

(iii) $M \in \mathbb{D}'$ implies that $M\backslash\{i\} \notin \mathbb{D}$ for any $i \in M$.

It is evident that one can choose at least one Nakamura family for any non-collegial family $\mathbb{D}$. In a previous paper [12] it has been shown that when $\mathbb{D}'$ is a Nakamura family then there exists a representation ϕ of $\mathbb{D}'$ of the following kind. First of all let $v(\mathbb{D}') = v(\mathbb{D}) = v$ and let $\mathbb{D}' = \{M_1,\ldots,M_v\}$. Let Δ be the standard simplex in $\mathbb{R}^{v-1}$ spanned by the set of vertices $Y = \{y_1,\ldots,y_v\}$. Let $N(\mathbb{D}') = \{i \in N : i \in M \text{ for some } M \in \mathbb{D}'\}$. A representation ϕ is a set-set correspondence

$$\phi : N(\mathbb{D}') \to \Delta$$

with the following properties.

(i) For each $M_j \in \mathbb{D}'$, $\phi(M_j) = F_j$ is the j^{th} face of Δ "opposite" the vertex y_j.

(ii) For each $i \in N(\mathbb{D}')$ let $\mathbb{D}_i = \{M \in \mathbb{D} : i \in M\}$. Then $\phi(\{i\})$ is the barycenter of the subsimplex $\bigcap_{M_k \in \mathbb{D}_i} \phi(M_k)$.

For example suppose $M_j \in \mathbb{D}'$ and consider $\mathbb{D}'' = \mathbb{D}'\backslash\{M_j\}$. Clearly $|\mathbb{D}''| = v - 1$ and so $K(\mathbb{D}'') \neq \Phi$. Thus there exists an individual whom we may label j such that $j \in K(\mathbb{D}'')$. By the definition $\phi(\{j\}) = \bigcap_{\mathbb{D}''} \phi(M_k)$. However, the faces $\phi(M_1),\ldots,\phi(M_{j-1})$, $\phi(M_{j+1}),\ldots,\phi(M_v)$ intersect precisely in the vertex y_j, opposite M_j and so $\phi(\{j\}) = y_j$. For convenience we call the set of individuals $V = \{1,\ldots,v\}$ the vertex group and note that each individual $j \in V$ is matched by ϕ to precisely the vertex y_j of Δ. For any subset Y' of $Y = \{y_1,\ldots,y_v\}$, let $\Delta(Y')$ be the subsimplex of Δ spanned by Y' and let $\theta(\Delta(Y'))$ be the barycenter of $\Delta(Y')$. It has been shown [12, Lemma 1] that for $i \in N(\mathbb{D}')$ it is the case that when $\phi(\{i\}) = \theta(\Delta(Y'))$ then the set Y' is characterized by $y_j \in Y'$ iff $i \notin M_j$ and $y_j \in Y\backslash Y'$ iff $i \in M_j$.

Example 1: To illustrate, suppose $N = \{1,2,3,4,5,6\}$ and $\mathbb{D} = \{M_1,M_2,M_3,M_4,M_5\}$ where $M_1 = \{2,3,4\}$, $M_2 = \{1,3,4\}$, $M_3 = \{1,2,4,5\}$, $M_4 = \{1,2,3,5\}$ and $M_5 = \{2,3,4,6\}$. Obviously $v(\mathbb{D}) = 4$. A Nakamura family is $\mathbb{D}' = \{M_1,M_2,M_3,M_4\}$. Note that $K(\mathbb{D}'\backslash\{M_j\}) = \{j\}$ for $j = 1,\ldots,4$ and so the vertex group is $\{1,2,3,4\}$. Let Δ be the simplex in $\mathbb{R}^3$ spanned by the four vertices $\{y_1,y_2,y_3,y_4\}$ and define $\phi(\{j\}) = y_j$ for $j = 1,\ldots,4$. Moreover $\mathbb{D}_5 = \{M_3,M_4\} = \mathbb{D}'\backslash\{M_1\}\backslash\{M_2\}$ and so, following the definition of ϕ, we let $\phi(\{5\})$ be the barycenter of the subsimplex spanned by (y_1,y_2). This is illustrated in Figure 1.

[Figure 1 here]

See [12] for full details of the construction.

Proof of Theorem 1(i): Let $v(\sigma) = v$ be the Nakamura number for σ, and let $\mathbb{D}'$ be a Nakamura family for $\mathbb{D}_\sigma$. By assumption $|W| \geq v$ and so there exists a subset $X = \{x_1,\ldots,x_v\}$ of W. Adopt the convention that $x_{t+v} = x_t$ for any integer $t \geq 0$. Let ϕ be a representation of $\mathbb{D}'$ with respect to Δ in $\mathbb{R}^{v-1}$, and let $Y = \{y_1,\ldots,y_v\}$ be the set of vertices of Δ. For each $j = 1,\ldots,v$ let

$\phi^{-1}(\{y_j\}) = \{i \in N(\mathbb{D}') : \phi(\{i\}) = \{y_j\}\}$ and to each $i \in \phi^{-1}(\{y_j\})$ assign the acyclic preference relation P_j on X which is given by

$$x_j P_j x_{j+1} P_j \cdots x_v P_j x_1 \cdots P_j x_{j-1}.$$

Now consider any subset Y' of Y with $|Y'| \geq 2$. To any individual $i \in N(\mathbb{D}')$ with $\phi(\{i\}) = \theta(\Delta(Y'))$ assign the acyclic preference profile $\bigwedge_j P_j$ where the meet is taken over all j such that $y_j \in Y'$. Now for each $M_j \in \mathbb{D}'$, we note that

$$\phi(M_j) = \Delta(Y_j) \text{ where } Y_j = Y\backslash\{y_j\}.$$

Thus if $i \in \phi^{-1}(\{y_k\})$ for $y_k \in Y_j$ then $x_{j-1}P_i x_j$. Moreover, if $i \in M_j$ then $\phi(\{i\}) \in \Delta(Y_j)$ and so $x_{j-1}P_i x_j$. Note that by the convention we have adopted $x_0 \equiv x_v$ and so $x_v P_i x_1$ for all $i \in M_1$. We have thus constructed a profile for the society $N(\mathbb{D}')$ on X. Extend this profile over W in the obvious way by defining $xP_i y$ whenever $x \in X$ and $y \in W\backslash X$ for each $j \in N(\mathbb{D}')$. To each individual $i \in N\backslash N(\mathbb{D}')$ assign an arbitrary but acyclic preference relation. Let P be the acyclic profile for N on W so constructed. From the construction it is the case that for $j = 2,\ldots,v$ we obtain $x_{j-1}P_i x_j$ for all $i \in M_j$. Moreover, $x_v P_i x_1$ for all $i \in M_1$. Since $\{M_1,\ldots,M_v\} \in \mathbb{D}_\sigma$ we obtain

$$x_v\sigma(P)x_1\sigma(P)x_2 \cdots x_{v-1}\sigma(P)x_v.$$

Thus $\sigma(P)$ is cyclic. By the construction it is evident that $GO(\sigma,P) = \Phi$.

Q.E.D.

Example 2: To illustrate the method of proof, consider Example 1 again. Suppose that $|W| \geq 4$ and let $X = \{x_1, x_2, x_3, x_4\}$ be a subset of W. Assign acyclic preferences to the vertex group $\{1,2,3,4\}$ on X as follows.

$$x_1 P_1 x_2 P_1 x_3 P_1 x_4$$
$$x_2 P_2 x_3 P_2 x_4 P_2 x_1$$
$$x_3 P_3 x_4 P_3 x_1 P_3 x_2$$
$$x_4 P_4 x_1 P_4 x_2 P_4 x_3.$$

As we saw we define $\phi(\{5\}) = \Theta(\Delta(\{y_1, y_2\}))$ and thus we let $P_5 = P_1 \wedge P_2$. Hence P_5 is given by

$$x_2 P_5 x_3 P_5 x_4.$$

Since $x_1 P_1 x_2$ and not $(x_1 P_2 x_2)$ we may assign an arbitrary preference for x_1, x_2 to individual 5. As we expect, for all $i \in M_1 = \{2,3,4\}$ we obtain $x_4 P_i x_1$. Note also that individual 5 agrees with the other members of M_3 that x_2 is preferred to x_3, and he agrees with the members of M_4 that x_3 is preferred to x_4. When preferences are extended over W and individual 6 is allocated an arbitrary preference we obtain a cycle

$$x_4 \sigma(P) x_1 \sigma(P) x_2 \sigma(P) x_3 \sigma(P) x_4$$

on X.

When σ is a social preference function and $\mathbb{D}'$ a Nakamura subfamily of $\mathbb{D}_\sigma$, then a subprofile P for $N(\mathbb{D}')$ on a set X of $v(\sigma)$ alternatives of the kind constructed in the proof of Theorem 1(i) we may call a $\mathbb{D}'$-permutation profile or simply a permutation profile for σ.

Proof of Theorem 1(iii): Suppose that $|W| = w \leq v - 1$ where $v(\sigma) = v$ and σ is a voting rule. Suppose further that there exists a cycle $x_r \sigma(P) x_1 \sigma(P) \cdots x_r$ where $r \leq w$. Let $M_j = \{i \in N : x_{j-1} P_i x_j\}$ where again we adopt the convention that $x_t \equiv x_{t+r}$. Clearly $\mathbb{D} = \{M_1, \ldots, M_r\} \subset \mathbb{D}_\sigma$ since σ is a voting rule. But $|\mathbb{D}| \leq v - 1$ and so $K(\mathbb{D}) = M_1 \cap \cdots \cap M_r \neq \Phi$. Hence there exists $i \in K(\mathbb{D})$ such that

$$x_r P_i x_1 P_i \cdots P_i x_r$$

Thus contradicts the assumption that $P_i \in A(W)$ for all $i \in N$. Thus $P \in A(W)^N$ implies there can exist no $\sigma(P)$-cycle on W.

Q.E.D.

Proof of Theorem 1(ii): By the third part of Theorem 1, if $|W| = v(\sigma) = v$, then there can exist no $\sigma(P)$ cycle on a proper subset of W. Thus if $GC(\sigma,P)$ is non-empty it must be the case that $GC(\sigma,P) = W$. Let $x_v \sigma(P) x_1 \cdots \sigma(P) x_v$. Again for $j = 1, \ldots, v$ let $M_j = \{i \in N : x_{j-1} P_i x_j\}$ and let $\mathbb{D}' = \{M_1, \ldots, M_v\}$. For each $j = 1, \ldots, v$ it is the case that $|\mathbb{D}' \backslash \{M_j\}| \leq v(\sigma) - 1$ and thus $K(\mathbb{D}' \backslash \{M_j\}) \neq \Phi$. After relabelling let $\{j\} = K(\mathbb{D}' \backslash \{M_j\})$ so that $x_j P_j x_{j+1} \cdots P_j x_{j-1}$. However, the preference for $\{j+1\}$ satisfies

$$x_{j+1} P_{j+1} x_{j+2} \cdots P_{j+1} x_j$$

and so not $(x_j P_{j+1} x_{j+1})$. Thus it is not the case that $x_j P_i x_{j+1}$ for all $i \varepsilon N$. Hence x_j and x_{j+1} are "pareto indifferent." Clearly $GO(N,P) = W$. Since $GC(\sigma,P) = W$ we obtain $GO(N,P) = GC(\sigma,P)$

Q.E.D.

As a further illustration of the theorem, consider the case where σ is a q-rule with $q = n - 1$. Then $v(n,n-1) = n - 2$ and so $v(\sigma) = n$. Thus the vertex group $V = \{1,\dots,n\}$ and a permutation profile is one based on permutations of the preference $x_1P_1x_2 \cdots P_1x_n$.

In a previous paper Ferejohn, Grether and McKelvey [3] showed that an (n - 1)-rule could be "manipulated" as long as $|W| \geq n - 1$. In the next section we shall extend their result to the case of a general rule.

4. MANIPULATION OF CHOICE FUNCTIONS

The existence of a permutation preference profile, of the kind constructed in the previous section, essentially means that a particular choice mechanism C can be <u>manipulated</u>.

The general idea is to suppose that the choice procedure is <u>implementable</u> in the sense that the outcomes selected by the choice procedure result from the individuals in the society selecting preference relations to submit to the choice procedure. These preference relations need not be "sincere" or truthful, but are in an appropriate sense optimal for the individuals in terms of their truthful preferences. An "implementable" choice procedure will then the <u>monotonic</u>. However, the existence of a permutation profile for σ

means that any choice mechanism which is compatible with the voting rule, σ, cannot be monotonic and thus cannot be implementable. Full details can be found in [3]. Here we briefly outline the proof that the existence of a permutation profile means the choice mechanism is not monotonic.

<u>Definition 2</u>:

(i) A <u>choice function</u> C is a mechanism which assigns to any set W and any preference profile $P \varepsilon A(W)^N$ a non-empty subset $C(W,P)$ of W.

(ii) The choice function, C, is <u>compatible</u> with a social preference function, σ, on W iff C satisfies the following property: for any $x \varepsilon W$, $P \varepsilon A(W)^N$ and $M \varepsilon \mathbb{D}_\sigma$ if it is the case that for each $i \varepsilon M$ there exists no $y_i \varepsilon W$ with y_iP_ix then $\{x\} = C(W,P)$.

(iii) The choice function, C, is <u>non-collegial</u> iff C is compatible with a non-collegial social preference function, σ. In this case define the <u>Nakamura number</u> $v(C)$ of C to be $v(\sigma)$.

(iv) If $P \varepsilon A(W)^N$ then a <u>manipulation</u> P' of P by a coalition M is a profile $P' = (P_1,\dots,P_n) \varepsilon A(W)^N$ such that $P_i = P_i'$ for $i \notin M$ and $P_i' \neq P_i$ for some $i \varepsilon M$.

(v) If $x \varepsilon C(W,P)$, for $P \varepsilon A(W)^N$ then C is <u>manipulable by M at (x,P)</u> iff there exists a manipulation P' of P by M, with $\{x'\} = C(W,P')$ for some $x' \neq x$, where $x'P_ix$ for all $i \varepsilon M$.

(vi) Say C is <u>non-manipulable</u> if for no x, no $M \subset N$ and no $P \varepsilon A(W)^N$ is C manipulable by M at (x,P).

(vii) The choice function C is monotonic on W iff whenever $x \in W$ and $P, P' \in A(W)^N$ such that $x \in C(W,P)$ and for all $i \in N$ and all $y \in W\backslash\{x\}$ it is the case that $xR(P_i)y$ implies $xR(P_i')y$ then $x \in C(W,P')$.

If a choice function C cannot be manipulated then it must be monotonic [3]. See also [5, 6]. In Theorem 2 we show that if $|W| \geq v(C)$ then it is possible to construct a permutation profile for a social preference function, σ, for which C is compatible. Consequently whatever a choice is made by C then C is manipulable at that choice.

Theorem 2: Let C be a non-collegial choice function on a finite set W. If $|W| \geq v(C)$ then C cannot be monotonic and thus C must be manipulable.

Proof: Let σ be a social preference function with which C is compatible. By definition $v(C) = v(\sigma)$. Since σ is non-collegial, $v(\sigma) < \infty$. Let $v = v(\sigma)$ and let $X = \{x_1,\ldots,x_v\} \subset W$. By Theorem 1 there exists a profile $P \in A(W)^N$ such that P/X, the profile P restricted to X, is a permutation profile for σ on X. Moreover, for all $i \in N$, all $x \in X$ and all $y \in W\backslash X$ it is the case that xP_iy. Consider any alternative $x \in C(W,P)$, and suppose first of all that $x = x_j \in X$, say. As in Theorem 1(i), there exists a coalition $M_j \in \mathbb{D}_\sigma$ such that each member, i, of M_j has a preference of the form

$$x_kR_ix_{k+1} \cdots x_{j-1}P_ix_jR_i \cdots x_{k-1}.$$

(Here R_i stands for weak preference of individual i). Let P' be the manipulation of P by M obtained by defining $x_{j-1}P_i'y$ for all $y \in W\backslash\{x_{j-1}\}$ all $i \in M$, but leaving all other preferences unchanged. Since $P \in A(W)^N$, this ensures that $P' \in A(W)^N$. Moreover, since C is σ-compatible, $\{x_{j-1}\} = C(W,P')$. Furthermore, since $x_{j-1}P_ix_j$, for all $i \in M_j$, C is manipulable at (x_j,P). Suppose now that C is monotonic. For all $i \in N$ it is the case that

$$\forall y \in W\backslash\{x_j\}, x_jR(P_i)y \Rightarrow x_jR(P_i')y.$$

By monotonicity, $x_j \in C(W,P')$. But $\{x_{j-1}\} = C(W,P')$. Hence C cannot be monotonic. In similar fashion, if $x \in W\backslash X$, then C is manipulable by each coalition $M_1,\ldots,M_v$ at (x,P). Again C cannot be monotonic. Thus there exists some $M \in \mathbb{D}_\sigma$ such that C is manipulable by M.

Q.E.D.

Corollary 1: If W is finite with $|W| \geq n$ and $n \geq 3$ then for no monotonic choice function C, does there exist a non-collegial voting rule σ, such that C is compatible with σ.

Proof: For any non-collegial voting rule, σ, it is the case that $v(\sigma) \leq n$ [10]. Thus if $|W| \geq n$ and C is compatible with a non-collegial voting rule σ, then by Theorem 2, C must be non-monotonic and hence manipulable.

Q.E.D.

Ferejohn, Grether and McKelvey [3] essentially obtained Theorem 2 in the case that σ was a q-rule with $q = n - 1$. In this case they said that a choice function that was compatible with σ was minimally democratic. They then showed that a minimally democratic choice function could be neither monotonic nor implementable (or non-manipulable) whenever $|W| \geq n$.

Example 3: Let P be the permutation preference profile on W constructed in Example 2. Let C be a choice function compatible with σ, and suppose, for purposes of illustration that $x_4 \in C(W,p)$. Note that each member of the coalition $M_4 = \{1,2,3,5\}$ prefers x_3 to x_4. Let P′ be the manipulation by M_4 of P which is given by

$$x_3 P_1' x_1 P_1' x_2 P_1' x_4$$
$$x_3 P_2' x_2 P_2' x_4 P_2' x_1$$
$$x_3 P_2' x_4 P_3' x_1 P_3' x_2$$
$$x_3 P_5' x_2 P_5' x_4 P_5' x_1.$$

let $P_4 = P_4'$ and $P_6 = P_6'$. Since $M_4 = \{1,2,3,5\} \in \mathbb{D}_\sigma$, $\{x_3\} = C(W,P')$. In identical fashion, whichever alternative is selected by the choice function, one of the four decisive coalitions may manipulate P to its advantage.

Consider the case with majority rule with n = 4. In this case the Nakamura number is 4. However, all other majority rules have Nakamura number 3. Thus majoritarian rules are essentially manipulable even with three alternatives. Peleg [8] has previously shown that if σ_q is a q-rule then there exists a non-manipulable choice function C compatible with σ_q on W iff $q > \left[\frac{w-1}{w}\right]n$ where

$w = |W|$. We use Theorem 2 to extend Peleg's result to an arbitrary voting rule.

Corollary 2: If σ is a proper and non-collegial voting rule as a finite set of alternatives, W, with Nakamura number $v(\sigma)$ then there exists a non-manipulable choice function C which is compatible with σ iff $|W| \leq v(\sigma) - 1$.

Proof: The necessity of the constraint $|W| \leq v(\sigma) - 1$ follows from Theorem 2. To show sufficiency define the mechanism C_σ by

$$C_\sigma(P) = GO(\sigma,W,N,P).$$

Clearly C_σ is compatible with σ. Moreover, by Theorem 1, if $|W| \leq v(\sigma) - 1$ then $GO(W,P) \neq \Phi$ for all $P \in A(W)^N$ and so C_σ is a choice function. Then by Peleg's method of proof [8, Theo. 4.3] C_σ is non-manipulable.

Q.E.D.

In the earlier paper [12] it was shown that if W is a convex subset of $\mathbb{R}^W$ of dimension at least $v(\sigma) - 1$ then it is always possible to construct an acyclic profile P of convex preferences on W such that the core $GO(\sigma,P)$ is empty and $GC(\sigma,P)$ is non-empty. An easy extension of that result can be used to show that the profile P has the following property: there exists a subset $X = \{x_1,\ldots,x_v\}$ of alternatives in W such that P, when restricted to X, is a permutation profile for a Nakamura subfamily of $\mathbb{D}_\sigma$. Indeed since the argument is "local" the argument is valid when W is a smooth manifold [12, Cor.

1]. Conversely, if $\dim(W) \leq v(\sigma) - 2$ then for convex preferences the optima set must be non-empty [11] and "local" permutation cycles cannot exist [10]. We may therefore state the following without proof.

Corollary 3: Let σ be a non-collegial voting rule. If W is a smooth manifold of dimension $\dim(W) \geq v(\sigma) - 1$ then there exists no σ-compatible non-manipulable choice function on W. On the other hand if W is a compact, convex subset of $\mathbb{R}^w$ with $w \leq v(\sigma) - 2$ then there exists a choice function C, defined on the set of all convex profiles on W, such that C is σ-compatible and non-manipulable.

Since $v(\sigma) = 3$ generally for majority rule it is the case that when the "policy space" W is a manifold of dimension at least two then essentially no choice function can be non-manipulable and compatible with majority rule [see 3, Theo. 2].

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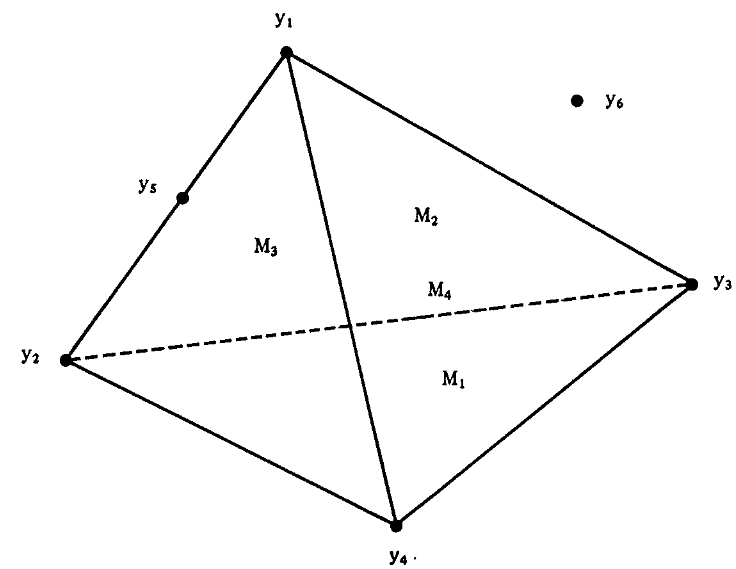

Figure 1: The representation in $\mathbb{R}^3$ of a voting rule with Nakamura number equal to four.